SLIGHTLY TWISTED JETS IN A COCURRENT FLOW

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Integration is used to provide an approximate solution for the problem of slightly twisted jets and wakes in a cocurrent flow for the case of laminar and turbulent flows of an incompressible liquid. A method involving the asymptotic thickness of the jet is used to achieve a solution to the problem for the case of a turbulent wake behind a rotating body.

The problem of propagating a twisted axial jet in an unbounded space has been considered by a number of authors (for example, [1-4]).

In this paper we present an approximate integral method of calculating the main segment of a slightly twisted jet in a cocurrent flow. No limitations are imposed on the relationship between the velocity u_δ outside the jet and the velocity u_0 within the nozzle. When $m = u_\delta/u_0 < 1$, we have the case of a jet in a cocurrent flow (the condition $m = 0$ corresponds to the special case of an immersed jet). Cases in which $m > 1$ come about in the wake behind a body or in the jet in a cocurrent flow. The following method is based on results obtained in [5] for plane and axisymmetric jets.

The integration results are compared with those obtained by other investigators, and for the case of a turbulent wake behind a rotating body the comparison is with the asymptotic jet-thickness solution presented below.

The equations of motion for slightly twisted axial flow in cocurrent flow can be written in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \mp\frac{1}{\rho r}\frac{\partial (r\tau_1)}{\partial r}, \tag{1}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial r} + \frac{vw}{r} = \frac{1}{\rho r^2}\frac{\partial (r^2\tau_2)}{\partial r}, \tag{2}$$

$$\frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial r} = 0. \tag{3}$$

Here and beyond, the upper sign corresponds to the wake and to the jet in the cocurrent flow when $m > 1$, the lower sign corresponds to the jet in the cocurrent flow when $m < 1$.

In the equations of motion (1)-(3), with only slight twisting it was assumed that $\operatorname{grad} p = 0$.

To isolate the nonzero solutions of the system of equations (1)-(3), we introduce the two invariants

$$I = 2\pi\rho\int_0^\delta u(u - u_\delta)\,r\,dr, \tag{4}$$

$$M = 2\pi\rho\int_0^\delta uwr^2\,dr. \tag{5}$$

Let us now present the profiles of the components for the frictional stress across the jet in the form of polynomials of degree r [6]:

$$\tau_i = \sum_0 A_{in}r^n \qquad (i = 1,\ 2). \tag{6}$$

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The coefficients of the polynomials in (6) are determined from (1)-(3) and from the boundary conditions at the symmetry axis and at the outside boundary of the jet:

$$r=0: \quad \tau_i=0, \quad \frac{\partial \tau_1}{\partial r} = \mp \rho u_m u'_m - \left(\frac{\tau_1}{r}\right)_{r=0},$$
$$\frac{\partial \tau_2}{\partial r} = -2\left(\frac{\tau_2}{r}\right)_{r=0}; \tag{7}$$
$$r=\delta: \quad \tau_i = \frac{\partial \tau_i}{\partial r} = 0.$$

Substituting (7) into (6), we have

$$\tau_1 = \left[\mp \rho u_m u'_m - \left(\frac{\tau_1}{r}\right)_0\right] \delta r_0 (1-r_0)^2, \tag{8}$$

$$\tau_2 = -2\left(\frac{\tau_2}{r}\right)_0 \delta r_0 (1-r_0)^2. \tag{9}$$

Relationships (8) and (9) are valid for both laminar and turbulent flow regimes.

Laminar Jet Flows. Here

$$\text{a) } \tau_1 = \mp \mu \frac{\partial u}{\partial r}, \quad \text{b) } \tau_2 = \mu r \frac{\partial}{\partial r}\left(\frac{w}{r}\right). \tag{10}$$

It follows from (1)-(5) and (8)-(10) that in the assumption of slight twisting the flow characteristics in the axial direction (u and δ) are independent of the twisting (or of the velocity component w).

Having substituted (8) into (10a), having integrated the latter, and having determined the integration constant from the condition $r_0 = 0$ and $u = u_m$, we have

$$u - u_m = \mp \frac{\delta}{12\mu}\left[\mp \rho u_m u'_m - \left(\frac{\tau_1}{r}\right)_0\right](6r_0^2 - 8r_0^3 + 3r_0^4). \tag{11}$$

When $r_0 = 1$, $u = u_\delta$ and it therefore follows from (11) that

$$u_\delta - u_m = \mp \frac{\delta}{12\mu}\left[\mp \rho u_m u'_m - \left(\frac{\tau_1}{r}\right)_0\right]. \tag{12}$$

Consequently, the profile of the axial velocity component from (1) and (12) assumes the form

$$u^0 = \frac{u_1}{u_{1m}} = 1 - 6r_0^2 + 8r_0^3 - 3r_0^4 \tag{13}$$

or

$$u = u_\delta \mp u_{1m}(1 - 6r_0^2 + 8r_0^3 - 3r_0^4). \tag{14}$$

Using (10a) and (14), we determine the magnitude of $(\tau_1/r)_0$ in (12):

$$(\tau_1/r)_0 = -12\mu u_{1m}/\delta^2. \tag{15}$$

Substitution of (15) into (12) yields

$$dx = -\frac{\delta^2}{24\nu}\frac{1 \mp u^0_{1m}}{u^0_{1m}} du^0_{1m}. \tag{16}$$

On discharge of the jet from a point source when x = 0, $u^0_{1m} = \infty$, while for the case of a wake when x = 0, $u^0_{1m} = 1$.

To integrate (16) we have to establish the relationship between δ and u^0_{1m}. With a wake or a jet in a cocurrent flow, this relationship is obtained from the condition of constant momentum (4) in all of the lateral cross sections of the jet.

Since the velocity profiles (13) and condition (4) coincide with the data of [5], we will simply use the results from that reference, namely:

$$\delta = Z_{1;2} \Big/ \left| 2\sqrt{\pi} \left(\frac{1}{10} u^0_{1m} \mp \frac{11}{210} u^{0^2}_{1m} \right)^{1/2} \right|. \tag{17}$$

Substituting (17) into (16) and integrating, we have

$$x = \frac{5 u_\delta Z_{1;2}}{48 \pi \nu} \left[\frac{1}{u^0_{1m}} \mp \frac{10}{21} \ln \left| \frac{1}{u^0_{1m}} \mp \frac{11}{21} \right| + a_{1;2} \right], \tag{18}$$

where $a_1 = -1.352$ and $a_2 = 0.306$.

In calculating the wake for a sufficiently great distance from the coordinate origin, in (17) and (18) the square of the velocity u^0_{1m} is negligibly small in comparison with its first power. Here

$$\delta = \left(24 \frac{\nu}{u_\delta} \right)^{1/2} x^{1/2}, \tag{19}$$

$$u_{1m} = \frac{F}{4.8 \pi \mu} \frac{1}{x}. \tag{20}$$

In the case of an immersed jet when $u_\delta = 0$, it follows analogously that

$$\delta = 7.78 \sqrt{\pi} \left(\frac{\nu^2}{K_1} \right)^{1/2} x, \tag{21}$$

$$u_m = \frac{1}{2.52 \pi} \frac{K_1}{\nu} \frac{1}{x}. \tag{22}$$

The resulting formulas completely solve the problem of calculating the axisymmetric laminar jet or wake in a cocurrent flow without twisting.

Substituting (9) into (10b), integrating, and determining the integration constant from the condition $r_0 = 1$ and $w = 0$, we have

$$w = \frac{2}{3} \frac{\delta^2}{\mu} \left(\frac{\tau_2}{r} \right)_0 r_0 \left[1 - r_0 (3 - 3r_0 + r_0^2) \right]. \tag{23}$$

It is easy to prove that the maximum of the velocity component w will be found when $r_0 = 1/4$ and it is equal to

$$w_m = \frac{9}{128} \frac{\delta^2}{\mu} \left(\frac{\tau_2}{r} \right)_0. \tag{24}$$

From (23) and (24) we find

$$w^0 = \frac{w}{w_m} = \frac{256}{27} r_0 \left[1 - r_0 (3 - 3r_0 + r_0^2) \right]. \tag{25}$$

The relationship between w_m and x is determined from the condition of a constant angular momentum (5) at any cross section of the jet.

Substituting (14) and (25) into (5) and integrating, we obtain

$$M = \frac{128}{945} \pi \rho \delta^3 u_\delta^2 w^0_m \left(1 \mp \frac{23}{33} u^0_{1m} \right),$$

which yields

$$w^0_m = \frac{945}{128 \pi} \frac{M}{\rho u_\delta^2} \delta^{-3} \left(1 \mp \frac{23}{33} u^0_{1m} \right)^{-1}. \tag{26}$$

It follows from (17) and (26) that

$$w^0_m = \frac{945}{128} \sqrt{\pi} \frac{M}{\rho u_\delta^2 Z^3_{1;2}} \left(\frac{1}{10} u^0_{1m} \mp \frac{11}{210} u^{0^2}_{1m} \right)^{3/2} \left(1 \mp \frac{23}{33} u^0_{1m} \right)^{-1}. \tag{27}$$

From large values of x and square of the velocity u_{1m}^0 is negligibly small in comparison with its first power. Here from (27) and (20) we have

$$w_m = \frac{1}{15.95\pi} \frac{M}{\rho u_\delta^2} \left(\frac{u_\delta}{\nu x}\right)^{3/2}. \tag{28}$$

In the case of an immersed jet when $u_\delta, = 0$, in analogous fashion from (27) and (22) we find

$$w_m = \frac{1}{17.7\pi^{3/2}} \frac{K_2}{\nu} \left(\frac{K_1}{\nu^2}\right)^{1/2} \frac{1}{x^2}. \tag{29}$$

For comparison let us present the results obtained by other researchers at this time.

For the case of a wake at a rather substantial distance from the rotating body Kalashnikov [6] found

$$u_{1m} = \frac{F}{4\pi\mu} \frac{1}{x}, \tag{30}$$

$$w_m^0 = \frac{1}{18.6\pi} \frac{M}{\rho u_\delta^2} \left(\frac{u_\delta}{\nu x}\right)^{3/2}. \tag{31}$$

For a laminar slightly twisted jet the self-similar solutions [1, 3] yield

$$u_m = \frac{1}{2{,}67\pi} \frac{K_1}{\nu} \frac{1}{x}, \tag{32}$$

$$w_m = \frac{1}{17.4\pi^{3/2}} \frac{K_2}{\nu} \left(\frac{K_1}{\nu^2}\right)^{1/2} \frac{1}{x^2}. \tag{33}$$

We see from a comparison of (20) with (30), (22) with (32), (28) with (31), and (29) with (33) that the corresponding formulas for the proposed integral method differ from the formulas for the "asymptotic" solutions only in terms of the numerical coefficients. This difference is slight and ranges within the usual limits of variation for the theory of a layer of finite thickness and for the theory of the asymptotic layer.

Turbulent Jet Flows. For a turbulent flow regime the relationship between the frictional stress components and those of the averaged velocity are chosen as follows:

$$\tau_1 = \mp \varepsilon \frac{\partial u}{dr}, \tag{34}$$

$$\tau_2 = \varepsilon r \frac{\partial}{\partial r} \left(\frac{w}{r}\right). \tag{35}$$

Since we are dealing with the problem of jet flow involving limited twist, on the basis of the author's results [7] it may be assumed in approximate terms that

$$\varepsilon = \varepsilon_0. \tag{36}$$

The relationship for ε_0 is usually [5] taken in the form

$$\varepsilon_0 = \rho\varkappa\delta u_{1m}. \tag{37}$$

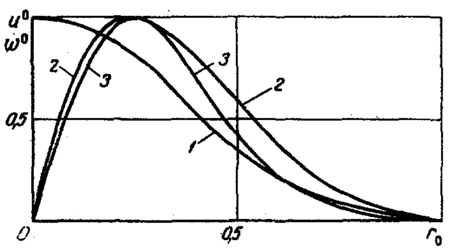
In (37) we can assume [8] that $\varkappa = 0.0097$.

It follows from (8), (9), and (34)-(37) that in the assumption of limited twist, as well as in the laminar regime, the flow characteristics in the axial direction (u and δ) are independent of twisting (or of w). Consequently, Eqs. (8), (34), and (37) form a system of equations for an axisymmetric untwisted jet flow. This problem is considered in [5]. We therefore present here the basic results from [5], which we will need in the following.

The profile of the velocity u, as before, has the form of (14), and the relationship between δ and u_{1m} is determined, again as before, by relationship (17). Finally, the variation of u_{1m}^0 with respect to x is described by the equation

$$x = \frac{Z_{1;2}}{24\sqrt{\pi\varkappa}} \left[\left(\frac{3.333}{u_{1m}^{0^2}} \mp \frac{6.507}{u_{1m}^0}\right) \sqrt{\frac{1}{10} u_{1m}^0 \mp \frac{11}{210} u_{1m}^{0^2} + b_{1;2}}\,\right], \tag{38}$$

where $b_1 = 0.692$ and $b_2 = -1.490$.



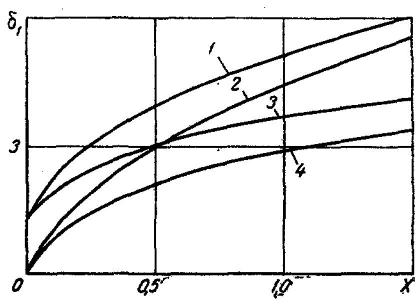
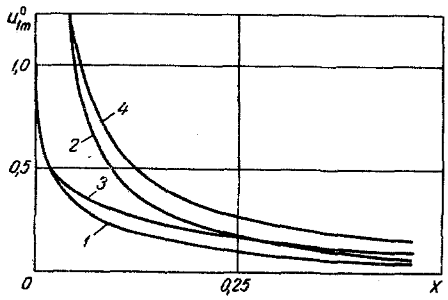


Fig. 1

Fig. 2

Fig. 1. Velocity profiles: 1) longitudinal velocity component u^0 calculated from (13); 2) circumferential velocity component w^0 calculated from (25); 3) circumferential velocity component w^0 from [6].

Fig. 2. Thickness of jet flow. Laminar regime: 1) wake behind body; 2) jet in cocurrent flow calculated from (17) and (18) $\delta_1 = \delta/Z_{1,2}$, $X = (\nu x)/(u\delta \cdot Z_{1,2}^{1}))$; turbulent regime: 3) wake behind body; 4) jet in cocurrent flow calculated from (17) and (38) $\delta_1 = \delta/Z_{1,2}$; $X = \varkappa x/Z_{1,2}$).



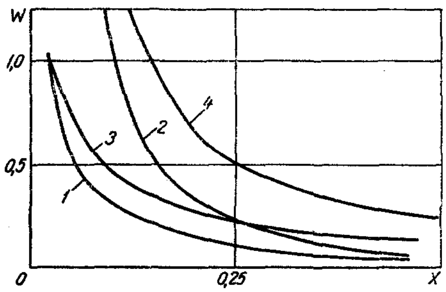


Fig. 3

Fig. 4

Fig. 3. Change of axial velocity component. Laminar regime: 1) wake behind body; 2) jet in cocurrent flow according to (18) $X = (\nu x)/(u\delta Z_{1,2}^{2}))$; turbulent regime: 3) wake behind body; 4) jet in cocurrent flow according to (38) $X = \varkappa x/Z_{1,2}$)

Fig. 4. Change of circumferential velocity component. Laminar regime: 1) wake behind a rotating body; 2) twisted jet in cocurrent flow according to (18) and (27) $W = (\rho u_\delta^2 Z_{1,2}^3/M)w_m^0$; $X = (\nu x)/(u\delta Z_{1,2}^2))$; turbulent regime: 3) wake behind a rotating body; 4) twisted jet in cocurrent flow according to (27) and (38) $W = (\rho u_\delta^2 Z_{1,2}^3/M)w_n^0$; $X = \varkappa x/Z_{1,2}$).

In calculating a wake at rather substantial distances from a body, we can use the following relationships (see the explanation preceding (19) and (20)):

$$\delta = 3.06\varkappa^{1/3}(c_x S x)^{1/3}, \tag{39}$$

$$u_{1m}^0 = \frac{0.099}{\varkappa^{2/3}}\left(\frac{c_x S}{x^2}\right)^{1/3}. \tag{40}$$

In the case of an immersed jet when $u_\delta = 0$, we analogously have

$$\delta = 24\varkappa x, \tag{41}$$

$$u_m = \frac{0.0726}{\varkappa}\left(\frac{I}{\rho}\right)^{1/2}\frac{1}{x}. \tag{42}$$

Let us not turn to the determination of the parameters which characterize twisting in turbulent jet flows.

It follows from (9) and (35)-(37) that the profile of the circumferential velocity component w is again determined by relationship (25), but in this case

$$w_m = \frac{9}{128}\left(\frac{\tau_2}{r}\right)_0 \frac{1}{\rho \varkappa u_{1m}}.$$

Since the profiles of the velocity components u and w for laminar and turbulent regimes are identical in form and since the relationship between δ and u_{1m} coincides (see (17)), for the turbulent regime formulas (26) and (27) remain valid.

In calculating the wake at rather large distances from a rotating body, on the basis of (40) we have

$$w_m = \frac{1}{7.72\pi}\,\frac{K_2}{u_\delta^2 Z_1^2}\,\frac{u_\delta}{\varkappa x}. \tag{43}$$

It follows from (42) that in the case of an immersed jet

$$w_m = \frac{1}{47.5\pi}\,\frac{K_2}{\varkappa^2 K_1^{1/2}}\,\frac{1}{x^2}. \tag{44}$$

A Turbulent Wake behind a Rotating Body. We present another solution for the problem of the turbulent wake behind a rotating body. The formulation of the problem coincides with that of the laminar wake [6]. We are dealing with an axisymmetric turbulent wake at a considerable distance behind a rotating body in the flow of an incompressible liquid.

The axial component of the velocity u is presented in the form

$$u = u_\delta + u'.$$

Let us assume that at a considerable distance from the body we have $u' \ll u_\delta$. The linearized system of equations for this problem from (1)-(3), (34), and (35) has the form

$$u_\delta \frac{\partial u'}{\partial x} = \varepsilon\left(\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\,\frac{\partial u'}{\partial r}\right) - \frac{1}{\rho}\,\frac{\partial p'}{\partial x}, \tag{45}$$

$$u_\delta \frac{\partial w}{\partial x} = \varepsilon\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} - \frac{w}{r^2}\right), \tag{46}$$

$$\frac{\partial p'}{\partial r} = \rho\,\frac{w^2}{r}, \tag{47}$$

$$\frac{\partial (ru')}{\partial x} + \frac{\partial (rv)}{\partial r} = 0. \tag{48}$$

As was pointed out above, for the case of slight twisting relationships (36) and (37) are valid. We will find the values of δ and u_{1m} from the problem of the untwisted wave [5] (see (39) and (40) of this paper). From (36), (37), (39), and (40) we therefore have

$$\varepsilon = 0.303\varkappa^{2/3} u_\delta (c_x S)^{2/3} x^{-1/3} = \nu_1 x^{-1/3}. \tag{49}$$

Considering (49), we come to the conclusion that Eqs. (45)-(48) are linear.

Introducing a new variable with the formulas

$$\eta = \frac{3}{2} x^{2/3}, \quad \frac{\partial}{\partial \eta} = x^{-1/3}\frac{\partial}{\partial x}, \tag{50}$$

we write system (45)-(48) in the form

$$u_\delta \frac{\partial u'}{\partial \eta} = \nu_1\left(\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\,\frac{\partial u'}{\partial r}\right) - \frac{1}{\rho}\,\frac{\partial p'}{\partial \eta}, \tag{51}$$

$$u_\delta \frac{\partial w}{\partial \eta} = \nu_1\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} - \frac{w}{r^2}\right), \tag{52}$$

$$\frac{\partial p'}{\partial r} = \rho\,\frac{w^2}{r}, \tag{53}$$

$$\frac{\partial (ru')}{\partial x} + \frac{\partial (rv)}{\partial r} = 0. \tag{54}$$

System (51)-(54) coincides with that derived in [6]. There we can also find the solutions for this system. Returning in the solutions to the variable x, we have

$$u_{1m}^{0} = \frac{0.088}{\varkappa^{2/3}} \left(\frac{c_x S}{x^2}\right)^{1/3}, \tag{55}$$

$$w_m = \frac{1}{5.71\pi} \frac{K_2}{u_\delta^2 Z_1^2} \frac{u_\delta}{\varkappa x}. \tag{56}$$

The formulas thus derived in this section provide for the complete solution of the problem of a slightly twisted turbulent wake in an incompressible liquid at a rather substantial distance from a rotating body.

For the case of a slightly twisted turbulent jet, the self-similar solution from [3] has the form

$$u_m = \frac{0.0633}{\varkappa} \left(\frac{I}{\rho}\right)^{1/2} \frac{1}{x}, \tag{57}$$

$$w_m = \frac{1}{57.5\pi} \frac{K_2}{\varkappa^2 K_1^2} \frac{1}{x^2}. \tag{58}$$

We see from a comparison of (40) with (55), (42) with (57), (43) with (56), and (44) with (58) that, as in the case of the laminar regime, the corresponding formulas of the proposed method differ from the formulas of the "asymptotic" solutions only in their numerical coefficients. This is a slight difference and falls within the range of conventional difference in the theory of a layer of finite thickness and in the theory of the asymptotic layer.

Figures 1-4 illustrate the basic results of this paper.

NOTATION

$A = M/16\pi\rho u_\delta$;	
$a_{1,2}$ and $b_{1,2}$	are constants;
c_x	is the coefficient of body drag;
F	is the body drag force;
I	is the momentum of the jet;
$K_1 = I/\rho$	is the kinematic jet momentum;
$K_2 = M/\rho$	is the kinematic angular momentum of the jet;
M	is the angular momentum of the jet;
$m = u_\delta\ u_0$;	
p'	is the excess pressure within the jet;
$r_0 = M$	is the area of the midsection of the body;
u, v, and w	are the components of the velocity in a cylindrical system of coordinates;
(x, r, θ)	
$u_m = u(0, x)$;	
$u_m' = du_m/dx$;	
u_δ	is the velocity of the uniform flow outside of the jet;
u_0	is the discharge velocity of the jet from the nozzle;
$u_1 = \mp(u - u_\delta)$;	
$u_{1m} = \mp(u_m - u_\delta)$;	
$u_{1m}^0 = u_{1m}/u_\delta$;	
$u' = u - u_\delta$;	
$u_0 = (u - u_\delta)/(u_m - u_\delta) = u_1/u_{1m}$;	
$w^0 = w/w_m$;	
$w_m^0 = w_m\ u_\delta$;	
$Z_1 = (c_x S)^{1/2}$;	
$Z_2 = [I/(\rho u_\delta^2)]^{1/2}$	
δ	is the thickness of the jet;
ε	is the coefficient of turbulent viscosity;
ε_0	is the value of ε for untwisted flow;

$\eta = (3/2)x^{2/3}$;

$\eta_1 = \nu_1 \eta / u_\delta$;

x — is an experimental constant;

μ — is the dynamic viscosity of the liquid;

$\nu = \mu/\rho$ — is the kinematic viscosity of the liquid;

$\nu_T = 0.303x^{2/3}$;

$\xi = r/\sqrt{2\nu_1}$;

ρ — is the density of the liquid;

τ_1 and τ_2 — are the components of the frictional stress in the direction of the x-axis in the circumferential direction, respectively.

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